1.4 – Inverses; Algebraic Properties of Matrices

Theorem 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid for matrices A, B, and C and scalars a, b, and c.

a) $A + B = B + A$ (commutative law for matrix addition)
b) $A + (B + C) = (A + B) + C = A + B + C$ (associative law for matrix addition)
c) $(AB)C = A(BC) = ABC$ (associative law for matrix multiplication)
d) $A(B + C) = AB + AC$ (left distributive law)
e) $(B + C)A = BA + CA$ (right distributive law)
f) $A(B - C) = AB - AC$
g) $(B - C)A = BA - CA$
h) $a(B + C) = aB + aC$
i) $a(B - C) = aB - aC$
j) $(a + b)C = aC + bC$
k) $(a - b)C = aC - bC$
l) $a(bC) = (ab)C$
m) $a(BC) = (aB)C = B(aC)$

Definition: In general, $AB \neq BA$. In the special cases where $AB = BA$, we say that A and B **commute**.

Definition: A **zero matrix**, denoted O, is a matrix whose entries are all zero.

Theorem 1.4.2 Properties of Zero Matrices
If c is a scalar, and if the sizes of the matrices A and O are such that the operations can be performed, then:
a) $A + O = O + A = A$
b) $A - O = A$
c) $A - A = A + (-A) = O$
d) $OA = O$
e) If $cA = O$, then $c = 0$ or $A = O$

Definition: **Identity matrices** are square matrices with 1's on the main diagonal and zeros everywhere else. They are denoted I or I_n if referencing the size, $n \times n$.

Theorem 1.4.3 If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has at least one row of zeros or R is the identity matrix I_n.

Definition: If A is a square matrix, and if there exists a matrix B of the same size for which $AB = BA = I$, then A is said to be **invertible** (or **nonsingular**) and B is called the **inverse** of A. If no such matrix B exists, then A is said to be **singular**. Because the inverse of a matrix A is unique, we will denote it using A^{-1}.

Theorem 1.4.4 Uniqueness of a Matrix Inverse
If B and C are both inverses of the matrix A, then $B = C$.

Theorem 1.4.5 The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

#8 Use Theorem 1.4.5 to compute the inverse.

$$\begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$$

Theorem 1.4.6 If A and B are invertible matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 1.4.7 Properties of Negative Exponents

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$.

b) A^n is invertible and $(A^n)^{-1} = A^{-n} = \left(A^{-1}\right)^n$.

c) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

#15 Use the given information to find A.

$$(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$$

#23 Find all values of a, b, c, and d (if any) for which the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ commute.

#25 Solve the system using an inverse matrix.

$$3x_1 - 2x_2 = -1$$

$$4x_1 + 5x_2 = 3$$

#39 Simplify the expression assuming that A, B, C and D are invertible.
$(AB)^{-1}\left(AC^{-1}\right)\left(D^{-1}C^{-1}\right)^{-1}D^{-1}$

Theorem 1.4.8 Properties of the Transpose
If the sizes of the matrices are such that the stated operations can be performed, then:
a) $\left(A^T\right)^T = A$
b) $(A+B)^T = A^T + B^T$
c) $(A-B)^T = A^T - B^T$
d) $(kA)^T = kA^T$
e) $(AB)^T = B^T A^T$

Theorem 1.4.9 If A is an invertible matrix, then A^T is also invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.